LTH 902

T-branes and Yukawa Couplings

Chan-Chi Chiou[1], Alon E. Faraggi[2] , Radu Tatar[3] and William Walters[4]

Division of Theoretical Physics, Department of Mathematical Sciences, The University of Liverpool, Liverpool, United Kingdom

Abstract

We consider various configurations of T-branes which are non-abelian bound states of branes and were recently introduced by Cecotti, Cordova, Heckman and Vafa. They are a refinement of the concept of monodromic branes featured in phenomenological F-theory models. We are particularly interested in the T-branes corresponding to Z_3 and Z_4 monodromies, which are used to break E_7 or E_8 gauge groups to $SU(5)_{GUT}$. Our results imply that the up-type and down-type Yukawa couplings for the breaking of E_7 are zero, whereas up-type and down-type Yukawa couplings, together with right handed neutrino Yukawas are non-zero for the case of the breaking of E_8. The dimension four proton decay mediating term is avoided in models with either E_7 or E_8 breaking.

January 2011

[1]e-mail: `chichi@liverpool.ac.uk`
[2]e-mail: `faraggi@liverpool.ac.uk`
[3]e-mail: `rtatar@liverpool.ac.uk`
[4]e-mail: `wrw23@liverpool.ac.uk`

Contents

1 Introduction

Recent studies have revealed that F-theory phenomenology can describe realistic features of particle phenomenology [1–3] (see also [4,5]). As compared to previous models for particle phenomenology involving D5 branes (type IIB) or D6 branes (type IIA), the F-theory approach localizes gauge fields on D7 branes wrapped on four cycles, matter on complex curves inside the four cycles and the Yukawa couplings at the intersection points of complex curves. The main observation of [1–3] was that all the fields are described in an eight-dimensional topological field theory giving rise to a four dimensional $N = 1$ supersymmetric field theory. The matter fields have Gaussian function on the directions normal to the matter curves so they are genuinely trapped on such curves. The F-theory trapping of matter curves can be nicely mapped into the matter curve restrictions appearing in heterotic strings [6].

The formulation in terms of the eight-dimensional topological field theory has led to a series of important developments which uncovered various aspects of F-theory phenomenology (for recent reviews on the subject see [7–9]). One issue first pointed out in [10] was related to the existence of novel features like branch cuts in models giving rise to relevant Yukawa couplings. This was due to the fact that, when studying such theories, the adjoint field Φ describing the displacement orthogonal to the worldvolume of the D7 branes was fixed to a background value $\langle\Phi\rangle$ taken to reside in the Cartan subalgebra. In order to achieve one heavy mass generation, a new concept needed to be introduced which is the seven brane monodromy. This means that, in order to deal with F-theory compactifications on a Calabi-Yau with generic complex structure, one needs to introduce branch cuts into the field theory and fields have to be twisted by Weyl reflections at the branch cuts. To get the Yukawa coupling in $SU(5)_{GUT}$, one needs to consider the breaking $E_6 \to SU(5)$ which involves a field Φ with the above branch cuts.

Very recently an important step has been taken towards a better understanding of this issue [11]. Instead of considering a diagonal $\langle\Phi\rangle$ with branch cuts, the authors of [11] have considered a theory without branch cuts but with a non-diagonal $\langle\Phi\rangle$ and denoted such non-diagonal case as seven-branes "T-branes". Some example were worked out for the case of T-branes with Z_2 or $Z_2 \times Z_2$ monodromy leading to breakings $E_6 \to SU(5)$ or $E_8 \to SU(5)$. The Yukawa couplings for the GUT group were computed with the help of a residue formula.

The present work extends the results of [11] to the case of Z_k monodromy T-branes when $k = 3$ and $k = 4$. The Z_3 monodromy T-branes are used to break the E_7 group to $SU(5)_{GUT} \times SU(3) \times U(1)$ whereas the Z_4 monodromy T-branes are used to breaking the E_8 group to $SU(5)_{GUT} \times SU(4) \times U(1)$. We consider the E_7 and E_8 breaking instead of just limiting to the E_6 breaking for several reasons. From [4] we know that we need the breaking of at least E_7 group in order to get two $\bar{5}$ representations which would correspond to $\bar{5}_M$ and $\bar{5}_H$. This would allow a down-type Yukawa coupling but prevent a proton decay term. The need for an E_7 gauge theory was also considered in [12] in order to avoid mixing between the resolution cycles.

On the other hand, we also know from [13] that the important feature of massive right handed neutrinos cannot be imposed at an E_7 point of enhancement. Our computations in the case of the breaking $E_7 \to SU(5)_{GUT}$, by using the residue results, show that both up-type and down-type Yukawa are zero for this case. We then proceed to the Z_4 monodromy model breaking E_8 to $SU(5)_{GUT}$. In this case all the matter curves have singularity at the origin but the residue results give non-zero Yukawa couplings, including for the Dirac neutrino mass term.

Another important issue for any phenomenological model is proton longevity. In addition to the matter–matter–Higgs couplings that are needed to generate the fermion mass spectrum, supersymmetric theories give rise to matter–matter–matter couplings that may yield dimension four and five baryon and lepton number violating operators. This is an intricate problem in Grand Unified Theories in general, and in string theories in particular. The reason being that while proton decay mediating operators must be adequately suppressed, Majorana neutrino masses require lepton number violation. These two key phenomenological constraints can be accommodated simultaneously, either by allowing lepton number, while forbidding baryon number, violating operators, or by using Dirac mass terms to generate the left–handed neutrino masses. However, in Grand Unified Theories that admit $SO(10)$ embedding of the Standard Model matter states only the former is possible due to lepton–quark mass relations that are dictated by the larger gauge symmetry. An appealing proposition to resolve this conundrum is the existence of gauged $U(1)$ symmetries that forbid the baryon number violating operators, while allowing lepton number violation. An example of a well know symmetry that partially does the job is that of gauged $U(1)_{B-L}$, which resides inside the GUT $SO(10)$. However, while $U(1)_{B-L}$ does forbid the dimension four lepton and baryon number violating operators, it does not forbid such dimension five operators, and is therefore not sufficient. Furthermore, $U(1)_{B-L}$ forbid the Majorana mass terms that are needed to generate light–neutrino masses. Hence, the desired symmetries must extend the $U(1)_{B-L}$ symmetry and reside outside $SO(10)$. In perturbative heterotic string theories the caveatted symmetries arise from the three generators in the Cartan sub–algebra of the observable E_8, and the existence of adequate phenomenological combinations has been explored in the literature [14–17]. In the present work we discuss the absence of proton decay within heterotic F–theory constructions, which naturally contain $U(1)$ symmetries residing outside $SO(10)$.

We start in section 2 with a brief discussion of the concept of monodromic branes introduced in [10] and the concept of T-branes introduced in [11]. We then continue with presenting the case of $Z_3, SU(3)$ and $Z_4, SU(4)$ backgrounds. Section 3 is the main section of this work, where we derive in detail the Yukawa couplings for the charged particles and for the singlets in the case of the Z_3 breaking of E_7 to $SU(5)_{GUT}$ and Z_4 breaking of E_8 to $SU(5)_{GUT}$. The details of the section 2 and section 3 computations are relegated to the Appendix where we also include some general Z_n results.

2 T-branes

2.1 Monodromic Branes

The usual way to deal with field theory on D-branes is to identify fields in the adjoint representations of gauge groups on branes with directions orthogonal to the D-branes. This has been used extensively for D3 branes probing singularities and D5 branes wrapped on resolution 2-cycles. The same method has been initiated for the case of D7 branes wrapped on 4-cycles for intersection of two D7-branes in [18], based on the findings of [19].

The results of [1–3] allow a further exploration of the results of [19]. Considering a four dimensional cycle S with complex coordinates $u_m,\ m = 1, 2$, the effective theory of zero modes along S can be described by an 8-dimensional field theory with four directions along S whose content is given in Table 1 where all the fields have their values in Lie algebra determined by the singularity along S.

The field $\overline{\Phi}_{\bar{m}\bar{n}} d\bar{u}_{\bar{m}} \wedge d\bar{u}_{\bar{n}}$ represents the transverse fluctuations of the D7 branes in the Calabi-Yau compactifications. The local geometry of the F-theory compactification can describe deformation of singularities which, in turn, can be related to describe matter and Yukawa couplings. In terms of the compactification of F-theory on

	vector	chiral	multiplet	anti-chiral	multiplet
Bosonic fields	A_μ	$\Phi_{mn} du_m \wedge du_n$	$A_{\bar{m}} d\bar{u}_{\bar{m}}$	$\overline{\Phi}_{\bar{m}\bar{n}} d\bar{u}_{\bar{m}} \wedge d\bar{u}_{\bar{n}}$	$A_m du_m$
Fermionic fields	η	$\chi_{mn} du_m \wedge du_n$	$\psi_{\bar{m}} d\bar{u}_{\bar{m}}$	$\bar{\chi}_{\bar{m}\bar{n}} d\bar{u}_{\bar{m}} \wedge d\bar{u}_{\bar{n}}$	$\bar{\psi}_m du_m$

Table 1: Field contents on S.

Calabi-Yau 4-folds, the vacuum expectation value for the field $\langle\Phi\rangle$ corresponds to local geometries of Calabi-Yau 4-fold and a mapping between the Calabi-Yau geometries and the values of $\langle\Phi\rangle$ has been used first in [18]. The approach involves a parametrization of a generic deformation of ADE singularities by $\mathfrak{h} \otimes C/W$ where $\mathfrak{h}$ is the Cartan subalgebra of the ADE algebra and W the Weyl group. On the other hand, for local Calabi–Yau geometry that are fibrations of deformation of a singularity over base space B, the local geometry is described using the above field theory with the background values for Φ lying in $\mathfrak{h} \otimes C$ and varying over B. The generic deformations can be easily mapped into the field theory quantities [19] if one ignores the difference between $\mathfrak{h} \otimes C/W$ and $\mathfrak{h} \otimes C$.

In the case of a non-compact complex curve B, the gauge group $\mathfrak{g}$ has one-rank larger than the singularity that remains over B after the deformation. The Weyl group is not important in this case and mapping is clear [18]. A more complicated case appears when B is a complex surface and the rank of the singularity group decreases by two.

The Weyl group becomes important and the mapping becomes less clear if the gauge group $\mathfrak{g}$ has a rank which is larger by at least two than the singularity group that remains over B after the deformation. The identification of [19] was extended in [10] to the case when generic deformations decrease the rank by two and the result was that the field $\langle\Phi\rangle$ had an unwanted feature of having branch cuts. As a simple example discussed in that paper, let us consider the deformation of the singularity $A_{N+1} \to A_{N-1}$ given by two parameters s_1 and s_2 :

$$Y^2 = X^2 + Z^N(Z^2 + s_1 Z + s_2), \tag{1}$$

and we consider the identification between the two parameters s_1 and s_2 (which are related to the non-zero values of $\langle\Phi\rangle$) and the coordinates of the complex surface (u_1, u_2) as

$$s_1 = 2u_1, \quad s_2 = u_2, \tag{2}$$

the non-zero part of $\langle\Phi\rangle$ has the values $(\sqrt{u_1^2 - u_2}, -\sqrt{u_1^2 - u_2})$ on the diagonal which acquire a minus sign around the branch locus $u_1^2 - u_2 = 0$.

One can try to generalize this model to the case with $A_{N+2} \to A_{N-1}$ given by two parameters s_1, s_2, s_3:

$$Y^2 = X^2 + Z^N(Z^3 + s_1 Z^2 + s_2 Z + s_3), \tag{3}$$

where s_1, s_2, s_3 would be directly related to the three non-zero diagonal entries for $\langle\Phi\rangle$. There is no isomorphism relation between s_i and u_i so a corresponding Z_3 model cannot be used to describe intersecting branes.

This effect showed that the seven-brane monodromy was a required ingredient in describing F-theory phenomenology and was subsequently developed along other directions [13, 20–23].

2.2 T-branes

The use of monodromic branes was based on the assumption that the field Φ is valued in the Cartan subalgebra. Very recently, the work of [11] used models where $\langle\Phi\rangle$ is upper triangular on some locus and such a configuration of seven-branes was denoted as T-branes, without the unwanted branch cuts on the field theory side. The difference between the T-branes and the intersecting branes lies in dealing with the spectral equation

$$P_\Phi(z) = \det(z - \Phi) = 0 \tag{4}$$

When Φ belongs to the Cartan subalgebra, the spectral equation becomes

$$\prod_i (z - \lambda_i) = 0 \tag{5}$$

where λ_i are the eigenvalues of Φ and they denote the directions of the intersecting branes. In case of non-diagonalizable Higgs fields, the spectral equation does not have a geometric interpretation and the intersecting branes picture does not hold, the monodromy group is now encoded in the form of the spectral equation. The paper [11] showed that the branch cuts are removed for the Z_2 model. One interesting aspect is that the T-brane model can consider more complicated $Z_n, n > 2$ cases like (3) which we now describe.

2.3 $SU(3)$

Let us consider the spectral equation for an $SU(3)$ field:

$$P_\Phi(z) = z^3 - x \tag{6}$$

for which there is a Z_3 monodromy. In the holomorphic gauge the Higgs field is:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}. \tag{7}$$

which is an intermediate case between a diagonal background and a nilpotent Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

We transform this to unitary gauge by using a positive diagonal matrix g with unit determinant:

$$g = \begin{pmatrix} e^{f_1} & 0 & 0 \\ 0 & e^{f_2} & 0 \\ 0 & 0 & e^{f_3} \end{pmatrix} \tag{9}$$

with the condition that $f_1 + f_2 + f_3 = 0$, and the f_a are real. Solving the D-term equation

$$\omega \wedge F_A + \frac{i}{2}\left[\Phi^\dagger, \Phi\right] = 0 \tag{10}$$

should give us the Toda equation

$$\Delta f_a = C_{ab} e^{f_b}, \tag{11}$$

where C_{ab} is the Cartan matrix of $SU(3)$ which is given by

$$C_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{12}$$

This presents an apparent problem, since there are three f_a's but C_{ab} is merely a 2x2 matrix. However this is because the unit determinant requirement of g means that there are only two linearly independent f_a's so the equation makes sense. We only end up with the required Toda equation if we take the three f_a's in g as specific linear combinations of the two linearly independent functions which we call h_a in Appendix.

As derived in the Appendix, the components for the unitary transformation for the nilpotent field Φ satisfy

$$\begin{aligned} \partial\overline{\partial} f_1 &= 2e^{f_1} - e^{f_2} \\ \partial\overline{\partial} f_2 &= -e^{f_1} + 2e^{f_2} \end{aligned} \tag{13}$$

and, for the general case

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}, \tag{14}$$

we get, for f_a depending only on r, the following two equations:

$$\begin{aligned} \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) f_1 &= r^{\frac{2}{3}}\left(2e^{f_1} - e^{-f_1-f_2} - e^{f_2}\right) \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) f_2 &= r^{\frac{2}{3}}\left(2e^{f_2} - e^{-f_1-f_2} - e^{f_1}\right). \end{aligned} \tag{15}$$

This set of equations generalizes the D-term equation

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right) = \frac{1}{2}\sinh(2f), \quad s = \frac{8}{3}r^{3/2} \tag{16}$$

obtained in [11] for the Z_2 T-branes which was a special instance of the Painleve III differential equation whose asymptotic behaviour was nicely mapped into the diagonalizable intersecting brane case for $r \to \infty$ and a nilpotent Higgs for $r \to 0$.

The asymptotic regions are represented by the case when r is either very small or very large. In our case, for r very small, the condition on g to be everywhere non-singular implies that near $r = 0$ the functions f_1, f_2 have logarithmic singularities and their exponentials approach non-zero constant matrices in the Cartan $U(1)^2$ of $SU(3)$. On the other hand, for large values of r we expect to get the case of intersecting branes obtained when the value of the flux F_A is zero. An explicit solutions for the f_1, f_2 should obey both limits for small and large r. We expect that a physically valid solution to exist and the configuration to be supersymmetric but a full solution involves generalizing the solution of Painleve III differential equation to the case of 2 functions.

2.3.1 Brane recombination

We consider infinitesimal perturbations to the holomorphic Higgs field of the form:

$$\varphi = \mathrm{ad}_{\Phi}(\xi) + h \tag{17}$$

where ξ is an arbitrary gauge transformation. Start with a $U(3)$ gauge theory, which can be thought of as corresponding to three superimposed D7-branes, and deform this theory using the $SU(3)$ Higgs vev

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}. \tag{18}$$

Consider the action of this field on an arbitrary gauge field

$$\xi = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{19}$$

as

$$\mathrm{ad}_{\Phi}(\xi) = [\Phi, \xi] = \begin{pmatrix} d - cx & e - a & f - b \\ g - fx & h - d & i - e \\ ax - ix & bx - g & cx - h \end{pmatrix}. \tag{20}$$

We then see that we can set certain components of the field h to zero. Firstly note that $\mathrm{ad}_{\Phi}(\xi)$ is traceless so we can remove two diagonal degrees of freedom from h. Then we can set h_{12} and h_{23} to zero using $e - a$ and $i - e$ respectively. However, after doing this, we do not have any freedom to set h_{31} to zero since

$$ax - ix = -x(e - a) - x(i - e) \tag{21}$$

Similarly we can set h_{13} and h_{21} to zero but then this fixes the gauge transform that can be made on h_{32} since

$$bx - g = -x(f - b) - (g - fx). \tag{22}$$

So the most general perturbation that can be made after gauge fixing is

$$\varphi = \begin{pmatrix} \frac{1}{3}\alpha(x, y) & 0 & 0 \\ 0 & \frac{1}{3}\alpha(x, y) & 0 \\ \gamma(x, y) & \beta(x, y) & \frac{1}{3}\alpha(x, y) \end{pmatrix}. \tag{23}$$

With this perturbation, the spectral equation is deformed as

$$z^3 - x \to \left(z - \frac{1}{3}\alpha(x, y)\right)\left(\left(z - \frac{1}{3}\alpha(x, y)\right)^2 - \beta(x, y)\right) - (x + \gamma(x, y)) \tag{24}$$

which, to first order in the perturbation, is

$$z^3 - z^2\alpha(x, y) - z\beta(x, y) - x - \gamma(x, y). \tag{25}$$

After changing coordinates to

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (z, y, P_\Phi(z)), \tag{26}$$

this becomes, in terms of the new brane worldvolume $\tilde{z} = 0$,

$$\tilde{z} - \left(\tilde{x}^2 \alpha\left(\tilde{x}^3, \tilde{y}\right) + \tilde{x}\beta\left(\tilde{x}^3, \tilde{y}\right) + \gamma\left(\tilde{x}^3, \tilde{y}\right)\right). \tag{27}$$

Hence the perturbations α, β and γ just make up the components of order $\tilde{x}^{3n+2}$, $\tilde{x}^{3n+1}$ and $\tilde{x}^{3n}$ of the Taylor expansion in $\tilde{x}$ of a single $U(1)$ field. This is interpreted as the three D7-branes recombining into a single D7-brane.

The Kahler metric on this brane can be determined by the pullback of the flat Kahler metric onto the brane. We start from the flat Kahler metric

$$\omega = \frac{i}{2}\left(dx \wedge d\overline{x} + dy \wedge d\overline{y} + dz \wedge d\overline{z}\right), \tag{28}$$

change to the new coordinates, and note that on the brane we have $\tilde{z} = P_\Phi(z) = z^3 - x = 0$, so that $x = z^3 = \tilde{x}^3$, we then have

$$\begin{aligned} dx &= 3\tilde{x}^2 d\tilde{x} \\ dy &= d\tilde{y} \\ dz &= d\tilde{x}. \end{aligned} \tag{29}$$

The Kahler form is

$$\omega = \frac{i}{2}\left(\left(1 + 9\left|\tilde{x}\right|^4\right) d\tilde{x} \wedge d\overline{\tilde{x}} + d\tilde{y} \wedge d\overline{\tilde{y}}\right), \tag{30}$$

so the recombined brane is curved, as in the $SU(2)$ case [11].

2.4 $SU(4)$

Let us now consider the spectral equation for an $SU(4)$ field:

$$P_\phi(z) = z^4 - x, \tag{31}$$

for which there is a Z_4 monodromy. In the holomorphic gauge the Higgs field becomes

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

which is an intermediate between a diagonal Higgs field and a nilpotent Higgs field in holomorphic gauge

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

As derived explicitly in the Appendix, we require that the components of the unitary transformation satisfy.

$$\begin{aligned} \partial\overline{\partial} f_1 &= 2e^{f_1} - e^{f_2} \\ \partial\overline{\partial} f_2 &= -e^{f_1} + 2e^{f_2} - e^{f_3} \\ \partial\overline{\partial} f_3 &= -e^{f_2} + 2e^{f_3}. \end{aligned} \tag{34}$$

which have the desired form

$$\partial\overline{\partial} f_a = C_{ab} e^{f_b}, \tag{35}$$

where C_{ab} is the $SU(4)$ Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{36}$$

For the general case, we proceed the same way as with $SU(3)$ (as derived in the Appendix) and get the equations for f_i as

$$\begin{aligned}
\partial\overline{\partial}f_1 &= r^{\frac{1}{2}}\left(2e^{f_1}-e^{f_2}-e^{-f_1-f_2-f_3}\right)\\
\partial\overline{\partial}f_2 &= r^{\frac{1}{2}}\left(-e^{f_1}+2e^{f_2}-e^{f_3}\right)\\
\partial\overline{\partial}f_3 &= r^{\frac{1}{2}}\left(-e^{f_2}+2e^{f_3}-e^{-f_1-f_2-f_3}\right). \qquad (37)
\end{aligned}$$

This set of equations is again similar to the one obtained in [11] for the Z_2 T-branes and we are unaware of any known solution for the set of differential equation for f_1, f_2, f_3.

2.4.1 Brane recombination

As was done in the $SU(3)$ case, we consider infinitesimal perturbations to the Higgs field and then see which can be gauged away to zero by a $U(4)$ gauge transformation. The result here turns out be that the most general perturbation after gauge fixing is

$$\varphi=\begin{pmatrix} \frac{1}{4}\alpha(x,y) & 0 & 0 & 0\\ 0 & \frac{1}{4}\alpha(x,y) & 0 & 0\\ 0 & 0 & \frac{1}{4}\alpha(x,y) & 0\\ \delta(x,y) & \gamma(x,y) & \beta(x,y) & \frac{1}{4}\alpha(x,y)\end{pmatrix}, \qquad (38)$$

which means that the spectral equation is now

$$\left(z-\frac{1}{4}\alpha(x,y)\right)\left(\left(z-\frac{1}{4}\alpha(x,y)\right)\left(\left(z-\frac{1}{4}\alpha(x,y)\right)^2-\beta(x,y)\right)-\gamma(x,y)\right)-(x+\delta(x,y)), \qquad (39)$$

expanding this to first order in the perturbation, one obtains

$$z^4-x-z^3\alpha(x,y)-z^2\beta(x,y)-z\gamma(x,y)-\delta(x,y). \qquad (40)$$

Then, changing coordinates to

$$(\tilde{x},\tilde{y},\tilde{z})=(z,y,P_\Phi(z)), \qquad (41)$$

(where here $P_\Phi(z)=z^4-x$ is the original spectral equation before the perturbations) this becomes

$$\tilde{z}-\left(\tilde{x}^3\alpha\left(\tilde{x}^4,\tilde{y}\right)-\tilde{x}^2\beta\left(\tilde{x}^4,\tilde{y}\right)-\tilde{x}\gamma\left(\tilde{x}^4,\tilde{y}\right)-\delta\left(\tilde{x}^4,\tilde{y}\right)\right). \qquad (42)$$

So, as with the $SU(3)$ case, the seemingly distinct fields are actually just components of a single field, so the effect of the Higgs vev is to recombine the four superimposed D7-branes with a $U(4)$ gauge group into a single D7-brane with a $U(1)$ gauge group.

Similarly to the $SU(4)$ case, the Kahler form on this recombined brane is given by

$$\omega=\frac{i}{2}\left(\left(1+16\left|\tilde{x}\right|^6\right)d\tilde{x}\wedge d\overline{\tilde{x}}+d\tilde{y}\wedge d\overline{\tilde{y}}\right), \qquad (43)$$

so this recombined brane is also curved.

3 GUT Models and $Z_k, k=3,4$ monodromy

This is the main section of our work and we use the backgrounds of the previous section together with the approach originated in [11] to derive formulas for various types of Yukawa couplings for $SU(5)$ F-theory GUT.

We are going to break either E_7 with an Z_3 T-brane and or E_8 with a Z_4 T-brane. Our conclusion is that the E_7 breaking gives rise to null up-type and down-type Yukawa couplings whereas the E_8 model gives rise to non-zero Yukawa couplings and also removes the proton decay term. We are also considering the singlet couplings (right handed neutrinos) and show that they are non-zero for the E_8 breaking. The Majorana masses are not allowed as we need the $U(1)$ symmetries.

3.1 Computation of Yukawa couplings for Localized modes

3.1.1 $E_7 \to SU(5) \times SU(3) \times U(1)$

Here we use an $SU(3) \times U(1)$ Higgs field which preserves an unbroken $SU(5)$:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix} \oplus (y)\,. \tag{44}$$

The adjoint of E_7 decomposes under this breaking as

$$\mathbf{133} \to (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{24},\mathbf{1})_0 \oplus \left(\overline{\mathbf{5}},\mathbf{3}\right)_{-2} \oplus \left(\mathbf{5},\overline{\mathbf{3}}\right)_2 \oplus \left(\overline{\mathbf{10}},\mathbf{3}\right)_1 \oplus \left(\mathbf{10},\overline{\mathbf{3}}\right)_{-1} \oplus (\mathbf{5},\mathbf{1})_{-3} \oplus \left(\overline{\mathbf{5}},\mathbf{1}\right)_3\,. \tag{45}$$

By noting that the required interaction terms are of the form

$$\mathbf{5}_H \cdot \mathbf{10}_M \cdot \mathbf{10}_M \quad \text{and} \quad \overline{\mathbf{5}}_H \cdot \overline{\mathbf{5}}_M \cdot \mathbf{10}_M, \tag{46}$$

and looking at the $U(1)$ charges in the decomposition, we see that we can identify the $\left(\mathbf{5},\overline{\mathbf{3}}\right)_2$ and the $\left(\overline{\mathbf{5}},\mathbf{3}\right)_{-2}$ as the $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$, and the $\left(\overline{\mathbf{5}},\mathbf{1}\right)_3$ as the $\overline{\mathbf{5}}_M$.

Looking at the $\overline{\mathbf{5}}_M$, we see that under the action of Φ the mode is simply multiplied by $3y$, which is obviously only invertible away from $y=0$ and so the torsion equation is solved with a matter curve $y=0$ and we have

$$\eta_{\overline{\mathbf{5}}_M} = \frac{1}{3}\varphi_{\overline{\mathbf{5}}_M}\,. \tag{47}$$

The $\mathbf{10}_M$ is in the antifundamental of the $SU(3)$

$$\varphi_{\mathbf{10}_M} = \begin{pmatrix} \varphi^1_{\mathbf{10}_M} \\ \varphi^2_{\mathbf{10}_M} \\ \varphi^3_{\mathbf{10}_M} \end{pmatrix}, \tag{48}$$

To see how this transforms under the Higgs field, we use the following basis:

$$\begin{pmatrix} e_2 \wedge e_3 \\ e_3 \wedge e_1 \\ e_1 \wedge e_2 \end{pmatrix}, \tag{49}$$

where the e_a span the fundamental of $SU(3)$ and the matrix acts on each component as

$$\Phi\left(e_a \wedge e_b\right) = \left(\Phi e_a\right) \wedge e_b + e_a \wedge \left(\Phi e_b\right). \tag{50}$$

So under a gauge transformation we have

$$\delta\varphi_{\mathbf{10}_M} = \begin{pmatrix} -2y & 0 & -x \\ -1 & -2y & 0 \\ 0 & -1 & -2y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{51}$$

Since this matrix is invertible when the determinant is non-zero, i.e. away from $-8y^3 - x = 0$, then $\varphi_{\mathbf{10}_M}$ is gauge equivalent to zero away from this locus, so clearly the matter curve is defined by

$$f = -8y^3 - x. \tag{52}$$

Notice that this matter curve is self-intersecting, as is required in order to have the coupling $\mathbf{5}_H \cdot \mathbf{10}_M \cdot \mathbf{10}_M$. On the matter curve, we can still set to zero the last two components of $\varphi_{\mathbf{10}_M}$, so we have

$$\varphi_{\mathbf{10}_M} = \begin{pmatrix} \varphi^1_{\mathbf{10}_M} \\ 0 \\ 0 \end{pmatrix}. \tag{53}$$

The torsion equation can be solved using the adjugate matrix:

$$\eta_{\mathbf{10}_M} = \begin{pmatrix} 4y^2 & x & -2xy \\ -2y & 4y^2 & x \\ 1 & -2y & 4y^2 \end{pmatrix} \begin{pmatrix} \varphi^1_{\mathbf{10}_M} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4y^2\varphi^1_{\mathbf{10}_M} \\ -2y\varphi^1_{\mathbf{10}_M} \\ \varphi^1_{\mathbf{10}_M} \end{pmatrix}. \tag{54}$$

The $\mathbf{5}_H$ transforms in the $\overline{\mathbf{3}}$ of $SU(3)$ and so we can use the result of the $\mathbf{10}_M$, just replacing y with $-2y$ because of the different $U(1)$ charge so the solution to the torsion equation is

$$\eta_{\mathbf{5}_H} = \begin{pmatrix} 16y^2 & x & 4xy \\ 4y & 16y^2 & x \\ 1 & 4y & 16y^2 \end{pmatrix} \begin{pmatrix} \varphi^1_{\mathbf{5}_H} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 16y^2\varphi^1_{\mathbf{5}_H} \\ 4y\varphi^1_{\mathbf{5}_H} \\ \varphi^1_{\mathbf{5}_H} \end{pmatrix}, \tag{55}$$

and the matter curve is given by $f = 64y^3 - x$.

The $\overline{\mathbf{5}}_H$ is in the fundamental of $SU(3)$ with a $U(1)$ charge of -2 so has the gauge transformation

$$\delta\varphi_{\overline{\mathbf{5}}_H} = \begin{pmatrix} -2y & 1 & 0 \\ 0 & -2y & 1 \\ x & 0 & -2y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{56}$$

giving a matter curve of $f = x - 8y^3$, and we can here set to zero the first two components of the triplet $\varphi_{\overline{\mathbf{5}}_H}$, and we have

$$\eta_{\overline{\mathbf{5}}_H} = \begin{pmatrix} 4y^2 & 2y & 1 \\ x & 4y^2 & 2y \\ 2xy & x & 4y^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \varphi^3_{\overline{\mathbf{5}}_H} \end{pmatrix} = \begin{pmatrix} \varphi^3_{\overline{\mathbf{5}}_H} \\ 2y\varphi^3_{\overline{\mathbf{5}}_H} \\ 4y^2\varphi^3_{\overline{\mathbf{5}}_H} \end{pmatrix}. \tag{57}$$

The $\mathbf{5}_H \cdot \mathbf{10}_M \cdot \mathbf{10}_M$ Yukawa is then

$$W_{\mathbf{5}_H\cdot\mathbf{10}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)} \left[\frac{\mathrm{Tr}\left(\left[\eta_{\mathbf{5}_H}, \eta_{\mathbf{10}_M}\right]\varphi_{\mathbf{10}_M}\right)}{(8y^3 + x)(64y^3 - x)} \right]. \tag{58}$$

Using the fact that the trace in the adjoint of $\mathfrak{e}_7$ (using i, j, k for $SU(5)$ indices and a, b, c for $SU(3)$ indices) is

$$\mathrm{Tr}\left(\left[t^a_{\mathbf{5}i}, t^b_{\mathbf{10}jk}\right] t^c_{\mathbf{10}lm}\right) \propto \epsilon_{ijklm}\epsilon^{abc} \tag{59}$$

and so the Yukawa becomes

$$W_{\mathbf{5}_H\cdot\mathbf{10}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)} \left[\frac{\epsilon_{ijklm} y \varphi^{1i}_{\mathbf{5}_H}\varphi^{1jk}_{\mathbf{10}_M}\varphi^{1lm}_{\mathbf{10}_M}}{(8y^3 + x)(64y^3 - x)} \right], \tag{60}$$

which is zero as it is holomorphic in y at $y = 0$. Note that in general any Yukawa coupling where all the matter curves are of the form

$$f = ay^n + bx \tag{61}$$

with $n \geq 2$ and a, b are arbitrary constants, will always be zero since the modes are always holomorphic and there will be no singularity at $y = 0$. This can be seen by noting that we can always move closer to the origin and of course therefore we can always choose $|ay^n| < |bx|$ and so we can factor out the bx and taylor expand to get a power series in y^n which is clearly holomorphic in y so giving a zero residue.

The other required Yukawa - $\overline{\mathbf{5}}_H \cdot \overline{\mathbf{5}}_M \cdot \mathbf{10}_M$, works out to be

$$W_{\overline{\mathbf{5}}_H\cdot\overline{\mathbf{5}}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)} \left[\frac{y^2\varphi^3_{\overline{\mathbf{5}}_H}\varphi_{\overline{\mathbf{5}}_M}\varphi^1_{\mathbf{10}_M}}{(y)(x - 8y^3)} \right] = \mathrm{Res}_{(0,0)} \left[\frac{y\varphi^3_{\overline{\mathbf{5}}_H}\varphi_{\overline{\mathbf{5}}_M}\varphi^1_{\mathbf{10}_M}}{(x - 8y^3)} \right] = 0. \tag{62}$$

We note that this particular $SU(5)$ GUT derived via breaking of the E_7 gauge group is not viable as neither of the required Yukawa couplings is present. This result is somewhat unexpected as it seems strange that the Yukawa couplings would vanish given that the symmetries allow for them and the matter curves intersect as required, we note that this is just one possible way of embedding $SU(5)$ into E_7.

We now turn our attention to the breaking via E_8.

3.1.2 $E_8 \to SU(5) \times SU(4) \times U(1)$

Consider an $SU(4) \times U(1)$ Higgs field which preserves an unbroken $SU(5)$. The Higgs is

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 \end{pmatrix} \oplus (y) \,. \tag{63}$$

The adjoint of E_8 decomposes as

$$\begin{aligned} \mathbf{248} \quad \to \quad & (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{4})_{-5} \oplus \left(\mathbf{1},\overline{\mathbf{4}}\right)_5 \oplus (\mathbf{1},\mathbf{15})_0 \oplus \left(\overline{\mathbf{5}},\mathbf{4}\right)_3 \oplus \left(\overline{\mathbf{5}},\mathbf{6}\right)_{-2} \oplus (\mathbf{10},\mathbf{1})_4 \oplus (\mathbf{10},\mathbf{4})_{-1} \\ & \oplus \left(\mathbf{5},\overline{\mathbf{4}}\right)_{-3} \oplus (\mathbf{5},\mathbf{6})_2 \oplus \left(\overline{\mathbf{10}},\mathbf{1}\right)_{-4} \oplus \left(\overline{\mathbf{10}},\overline{\mathbf{4}}\right)_1 \end{aligned} \tag{64}$$

Looking at the $U(1)$ charges and the required couplings we can identify the $(\mathbf{10},\mathbf{4})_{-1}$ as the $\mathbf{10}_M$, the $(\mathbf{5},\mathbf{6})_2$ as the $\mathbf{5}_H$, the $\left(\overline{\mathbf{5}},\mathbf{6}\right)_{-2}$ as the $\overline{\mathbf{5}}_H$ and the $\left(\overline{\mathbf{5}},\mathbf{4}\right)_3$ as the $\overline{\mathbf{5}}_M$.

Writing the $\mathbf{10}_M$ mode as

$$\varphi_{\mathbf{10}_M} = \begin{pmatrix} \varphi^1_{\mathbf{10}_M} \\ \varphi^2_{\mathbf{10}_M} \\ \varphi^3_{\mathbf{10}_M} \\ \varphi^4_{\mathbf{10}_M} \end{pmatrix}, \tag{65}$$

an $SU(4)$ gauge transformation acts on it as

$$\delta\varphi_{\mathbf{10}_M} = \begin{pmatrix} -y & 1 & 0 & 0 \\ 0 & -y & 1 & 0 \\ 0 & 0 & -y & 1 \\ x & 0 & 0 & -y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{66}$$

We can choose all but the last component of $\varphi_{\mathbf{10}_M}$ to be zero on the matter curve $y^4 - x = 0$. As with the E_7 model, the $\mathbf{10}_M$ curve is self-intersecting as required.

We can multiply by the adjugate matrix to solve the torsion equation and obtain the localised mode:

$$\eta_{\mathbf{10}_M} = \begin{pmatrix} -y^3 & -y^2 & -y & -1 \\ -x & -y^3 & -y^2 & -y \\ -xy & -x & -y^3 & -y^2 \\ -xy^2 & -xy & -x & -y^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi^4_{\mathbf{10}_M} \end{pmatrix} = \begin{pmatrix} -\varphi^4_{\mathbf{10}_M} \\ -y\varphi^4_{\mathbf{10}_M} \\ -y^2\varphi^4_{\mathbf{10}_M} \\ -y^3\varphi^4_{\mathbf{10}_M} \end{pmatrix}. \tag{67}$$

The $\overline{\mathbf{5}}_M$ is also in the fundamental of the $SU(4)$ and transforms just as the $\mathbf{10}_M$ but with a different $U(1)$ charge, so one just replaces y with $-3y$ in the above result, yielding

$$\eta_{\overline{\mathbf{5}}_M} = \begin{pmatrix} -\varphi^4_{\overline{\mathbf{5}}_M} \\ 3y\varphi^4_{\overline{\mathbf{5}}_M} \\ -9y^2\varphi^4_{\overline{\mathbf{5}}_M} \\ 27y^3\varphi^4_{\overline{\mathbf{5}}_M} \end{pmatrix}, \tag{68}$$

with a matter curve $f = 81y^4 - x$.

The $\mathbf{5}_H$, however, transforms in the $\mathbf{6}$ of $SU(4)$ and we write it in components as

$$\varphi_{\mathbf{5}_H} = \varphi^{ab}_{\mathbf{5}_H} e_a \wedge e_b, \tag{69}$$

where similar to the $SU(3)$ case above, e_a span the fundamental of $SU(4)$ and the basis elements $e_a \wedge e_b$ each transform under Φ as

$$\Phi\left(e_a \wedge e_b\right) = \left(\Phi e_a\right) \wedge e_b + e_a \wedge \left(\Phi e_b\right). \tag{70}$$

We write the elements of $\varphi_{\mathbf{5}_H}$ as a six dimensional vector with the following basis:

$$\begin{pmatrix} e_1 \wedge e_2 \\ e_1 \wedge e_3 \\ e_1 \wedge e_4 \\ e_2 \wedge e_3 \\ e_2 \wedge e_4 \\ e_3 \wedge e_4 \end{pmatrix}. \tag{71}$$

Under a gauge transformation, $\varphi_{\mathbf{5}_H}$ transforms as

$$\delta\varphi_{\mathbf{5}_H} = \begin{pmatrix} 4y & 1 & 0 & 0 & 0 & 0 \\ 0 & 4y & 1 & 1 & 0 & 0 \\ 0 & 0 & 4y & 0 & 1 & 0 \\ 0 & 0 & 0 & 4y & 1 & 0 \\ -x & 0 & 0 & 0 & 4y & 1 \\ 0 & -x & 0 & 0 & 0 & 4y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 4ya+b \\ 4yb+c+d \\ 4yc+e \\ 4yd+e \\ -xa+4ye+f \\ -xb+4yf \end{pmatrix}, \tag{72}$$

from which we can see that the matter curve has a factorisable form $64y^2\left(64y^4+x\right)=0$.

We also see that we can use this gauge transformation to set the 1st, 2nd, 5th and one of the 3rd or 4th components to zero, we choose to set the third to zero. Using the adjugate matrix we solve the torsion equation as

$$\begin{aligned} \eta_{\mathbf{5}_H} &= \begin{pmatrix} 8xy+1024y^5 & -256y^4 & 64y^3 & 64y^3 & -32y^2 & 8y \\ 32xy^2 & 1024y^5 & -256y^4 & -256y^4 & 128y^3 & -32y^2 \\ -64xy^3 & 32xy^2 & 8xy+1024y^5 & -8xy & -256y^4 & 64y^3 \\ -64xy^3 & 32xy^2 & -8xy & 8xy+1024y^5 & -256y^4 & 64y^3 \\ 256xy^4 & -128xy^3 & 32xy^2 & 32xy^2 & 1024y^5 & -256y^4 \\ 8x^2y & 256xy^4 & -64xy^3 & -64xy^3 & 32xy^2 & 8xy+1024y^5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi^{23}_{\mathbf{5}_H} \\ 0 \\ \varphi^{34}_{\mathbf{5}_H} \end{pmatrix} \\ &= \begin{pmatrix} 64y^3\varphi^{23}_{\mathbf{5}_H}+8y\varphi^{34}_{\mathbf{5}_H} \\ -256y^4\varphi^{23}_{\mathbf{5}_H}-32y^2\varphi^{34}_{\mathbf{5}_H} \\ -8xy\varphi^{23}_{\mathbf{5}_H}+64y^3\varphi^{34}_{\mathbf{5}_H} \\ \left(8xy+1024y^5\right)\varphi^{23}_{\mathbf{5}_H}+64y^3\varphi^{34}_{\mathbf{5}_H} \\ 32xy^2\varphi^{23}_{\mathbf{5}_H}-256y^4\varphi^{34}_{\mathbf{5}_H} \\ -64xy^3\varphi^{23}_{\mathbf{5}_H}+\left(8xy+1024y^5\right)\varphi^{34}_{\mathbf{5}_H} \end{pmatrix}. \end{aligned} \tag{73}$$

The $\overline{\mathbf{5}}_H$ transforms the same as the $\mathbf{5}_H$ under $SU(4)$, but with the opposite $U(1)$ charge, so it has the same matter curve of $f=64y^2\left(64y^4+x\right)$ and the localized mode is given by

$$\eta_{\overline{\mathbf{5}}_H} = \begin{pmatrix} -64y^3\varphi^{23}_{\overline{\mathbf{5}}_H}-8y\varphi^{34}_{\overline{\mathbf{5}}_H} \\ -256y^4\varphi^{23}_{\overline{\mathbf{5}}_H}-32y^2\varphi^{34}_{\overline{\mathbf{5}}_H} \\ 8xy\varphi^{23}_{\overline{\mathbf{5}}_H}-64y^3\varphi^{34}_{\overline{\mathbf{5}}_H} \\ -\left(8xy+1024y^5\right)\varphi^{23}_{\overline{\mathbf{5}}_H}-64y^3\varphi^{34}_{\overline{\mathbf{5}}_H} \\ 32xy^2\varphi^{23}_{\overline{\mathbf{5}}_H}-256y^4\varphi^{34}_{\overline{\mathbf{5}}_H} \\ 64xy^3\varphi^{23}_{\overline{\mathbf{5}}_H}-\left(8xy+1024y^5\right)\varphi^{34}_{\overline{\mathbf{5}}_H} \end{pmatrix}. \tag{74}$$

The $\mathbf{5}_H \cdot \mathbf{10}_M \cdot \mathbf{10}_M$ Yukawa is then

$$W_{\mathbf{5}_H\cdot\mathbf{10}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)}\left[\frac{\mathrm{Tr}\left(\left[\eta_{\mathbf{5}_H},\eta_{\mathbf{10}_M}\right]\varphi_{\mathbf{10}_M}\right)}{y^2\left(64y^4+x\right)\left(y^4-x\right)}\right]. \tag{75}$$

The trace in $\mathfrak{e}_8$, using i,j,k for $SU(5)$ indices and a,b,c for $SU(4)$ indices is

$$\mathrm{Tr}\left(\left[t^{ab}_{\mathbf{5}i},t^{c}_{\mathbf{10}jk}\right]t^{d}_{\mathbf{10}_{lm}}\right) \propto \epsilon_{ijklm}\epsilon^{abcd}. \tag{76}$$

So then the Yukawa becomes

$$
\begin{aligned}
W_{\mathbf{5}_H\cdot\mathbf{10}_M\cdot\mathbf{10}_M} &= \mathrm{Res}_{(0,0)}\left[\frac{\epsilon_{ijklm}\left(64y^3\varphi^{23i}_{\mathbf{5}_H}+8y\varphi^{34i}_{\mathbf{5}_H}\right)y^2\varphi^{4jk}_{\mathbf{10}_M}\varphi^{4lm}_{\mathbf{10}_M}}{\left(y^2\left(64y^4+x\right)\right)\left(y^4-x\right)}\right] \\
&+\mathrm{Res}_{(0,0)}\left[\frac{\epsilon_{ijklm}\left(256y^4\varphi^{23i}_{\mathbf{5}_H}+32y^2\varphi^{34i}_{\mathbf{5}_H}\right)y\varphi^{4jk}_{\mathbf{10}_M}\varphi^{4lm}_{\mathbf{10}_M}}{\left(y^2\left(64y^4+x\right)\right)\left(y^4-x\right)}\right] \\
&+\mathrm{Res}_{(0,0)}\left[\frac{\epsilon_{ijklm}\left(\left(8xy+1024y^5\right)\varphi^{23i}_{\mathbf{5}_H}+64y^3\varphi^{34i}_{\mathbf{5}_H}\right)\varphi^{4jk}_{\mathbf{10}_M}\varphi^{4lm}_{\mathbf{10}_M}}{\left(y^2\left(64y^4+x\right)\right)\left(y^4-x\right)}\right].
\end{aligned}
\tag{77}
$$

Which, after simplifying, becomes

$$
W_{\mathbf{5}_H\cdot\mathbf{10}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)}\left[\frac{\epsilon_{ijklm}\varphi^{23i}_{\mathbf{5}_H}\varphi^{4jk}_{\mathbf{10}_M}\varphi^{4lm}_{\mathbf{10}_M}}{(x)\,(y)}\right]. \tag{78}
$$

Using the trace result

$$
\mathrm{Tr}\left(\left[t^a_{\overline{\mathbf{5}}i}, t^b_{\mathbf{10}jk}\right]t^{cd}_{\overline{\mathbf{5}}l}\right) \propto \delta_{ij}\delta_{kl}\epsilon^{abcd}, \tag{79}
$$

the $\overline{\mathbf{5}}_H\cdot\overline{\mathbf{5}}_M\cdot\mathbf{10}_M$ Yukawa works out to be

$$
W_{\overline{\mathbf{5}}_H\cdot\overline{\mathbf{5}}_M\cdot\mathbf{10}_M} = \mathrm{Res}_{(0,0)}\left[\frac{\varphi^{23}_{\overline{\mathbf{5}}_H i}\varphi^{4}_{\overline{\mathbf{5}}_M j}\varphi^{4ij}_{\mathbf{10}_M}}{(x)\,(y)}\right]. \tag{80}
$$

As advertised in the Introduction, one important feature for the models with E_7 or E_8 breaking is the absence of the proton decay terms. For the E_8 case discussed in this subsection, the 4-dimensional proton decay mediating operators would be of the form $\overline{\mathbf{5}}_M\cdot\mathbf{10}_M\cdot\overline{\mathbf{5}}_M$ or $\mathbf{5}_M\cdot\mathbf{10}_M\cdot\mathbf{10}_M$. Without any further computations, we see from the definitions of the field charges that these terms are forbidden because they do have the allowed $U(1)$ charges, as discussed in [24].

3.2 Right-handed Neutrinos

Another important coupling is that of the right-handed neutrino, which is a singlet of $SU(5)$. It can couple in either Dirac or Majorana way. The Dirac scenario requires just one coupling of the form $\mathbf{5}_H\cdot\overline{\mathbf{5}}_M\cdot\mathbf{1}$. As considered in reference [5], the right handed neutrinos can be seen as complex structure deformations. If F-theory is compactified on X, a Calabi-Yau 4-fold, the $SU(5)_{GUT}$ singlet field in the singlet Yukawa coupling was considered to be related to fluctuations from the vacuum in $H^{1,2}(X)$ and the Yukawa coupling was calculated by an overlap integration

$$
\int_S \mathrm{tr}(\chi_{\mathbf{6}}\psi_{\mathbf{15}}\psi_{\mathbf{4}}) \tag{81}
$$

where $\mathbf{6},\mathbf{15},\mathbf{4}$ refer to the representations of the transverse $SU(4)$. If we identify the right handed neutrinos with the adjoint representation of the transverse group, then we cannot use the residue formula to compute its Yukawa coupling. This is expected as the right-handed neutrino is not localized on matter curve but corresponds to deformations of the complex structure.

On the other hand, the results of [5] were based on the fact that the field Φ is diagonal when the deformation of the complex structure of X of an F-theory compactification correspond to the the (2,0) forms in the Cartan part of the transverse group. By using the T-brane formalism, this consideration should be changed and one needs to rethink the issue of identifying the complex structure deformations. We leave this issue for a future publication.

4 Conclusions

In the present work we have presented some models of T-branes which correspond to brane configurations with Z_3 and Z_4 monodromies. These configurations have been used to break the E_7 and E_8 groups to the $SU(5)$ grand unification group. We used the residue formulas to compute Yukawa couplings for both $E_7\to SU(5)\times SU(3)\times U(1)$ and $E_8\to SU(5)\times SU(4)\times U(1)$.

There are two interesting directions which can be followed. The first one involves obtaining a solution to the differential equation for Z_3 and Z_4 background derived in our work. They should be generalizations of the Painleve III differential equations and would allow an explicit solution for the supersymmetric brane configurations. The second direction is to obtain an understanding of right handed neutrinos in the context of T-branes where one goes beyond the Cartan subalgebra. This would allow further insights into Yukawa couplings for right handed neutrinos.

This is a particularly relevant issue fo F-theory studies, due to the absence of adjoint and higher order scalar representations in perturbative heterotic constructions [25, 26], specifically, the absence of the **126** representation of $SO(10)$ in perturbative constructions indicates that the right-handed neutrino Majorana mass can only be generated by a vev of a Higgs field in the **16** representation, hence breaking lepton number by one unit. The interesting question therefore is whether the non-perturbative framework of F-theory offers some new possibilities.

Acknowledgements

We would like to thank Taizan Watari for important discussions over related subjects. This work is supported by the STFC under contract PP/D000416/1. We would also like to thank our referee for helpful suggestions.

A Appendix

A.1 Details of $SU(3)$ Computations

We present here the details of the computations for the SU(3) case corresponding to a Z_3 T-brane. Firstly we work out our new Higgs field in unitary gauge:

$$\Phi = g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & e^{f_1 - f_2} & 0 \\ 0 & 0 & e^{f_2 - f_3} \\ 0 & 0 & 0 \end{pmatrix}. \tag{82}$$

then the commutator part of the D-term equation which is

$$\left[\Phi^\dagger, \Phi\right] = \begin{pmatrix} -e^{2(f_1-f_2)} & 0 & 0 \\ 0 & e^{2(f_1-f_2)} - e^{2(f_2-f_3)} & 0 \\ 0 & 0 & e^{2(f_2-f_3)} \end{pmatrix}, \tag{83}$$

But the Toda equation requires only two independent function, we denote them by $e^{h_1}$and e^{h_2}, with

$$\begin{aligned} h_1 &= 2(f_1 - f_2) \\ h_2 &= 2(f_2 - f_3). \end{aligned} \tag{84}$$

Combined with $f_1 + f_2 + f_3 = 0$, we obtain

$$\begin{aligned} f_1 &= \frac{1}{6}(2h_1 + h_2) \\ f_2 &= \frac{1}{6}(h_2 - h_1) \\ f_3 &= \frac{1}{6}(-h_1 - 2h_2). \end{aligned} \tag{85}$$

Hence the required unitary transformation, reverting to f_a instead of h_a for the linearly independent functions, is

$$g = \begin{pmatrix} e^{\frac{1}{6}(2f_1+f_2)} & 0 & 0 \\ 0 & e^{\frac{1}{6}(f_2-f_1)} & 0 \\ 0 & 0 & e^{\frac{1}{6}(-f_1-2f_2)} \end{pmatrix}. \tag{86}$$

This gives a transformed Higgs field

$$\Phi = \begin{pmatrix} 0 & e^{\frac{1}{2}f_1} & 0 \\ 0 & 0 & e^{\frac{1}{2}f_2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{87}$$

and the commutator part of the D-term equation

$$\left[\Phi^\dagger, \Phi\right] = \begin{pmatrix} -e^{f_1} & 0 & 0 \\ 0 & e^{f_1} - e^{f_2} & 0 \\ 0 & 0 & e^{f_2} \end{pmatrix}. \tag{88}$$

The connection is

$$A^{0,1} = g\overline{\partial}g^{-1} = \frac{1}{6}\begin{pmatrix} -2\overline{\partial}f_1 - \overline{\partial}f_2 & 0 & 0 \\ 0 & \overline{\partial}f_1 - \overline{\partial}f_2 & 0 \\ 0 & 0 & \overline{\partial}f_1 + 2\overline{\partial}f_2 \end{pmatrix}, \tag{89}$$

and the D-term equations become

$$\begin{aligned} \frac{1}{3}\left(-2\partial\overline{\partial}f_1 - \partial\overline{\partial}f_2\right) &= -e^{f_1} \\ \frac{1}{3}\left(\partial\overline{\partial}f_1 - \partial\overline{\partial}f_2\right) &= e^{f_1} - e^{f_2} \\ \frac{1}{3}\left(\partial\overline{\partial}f_1 + 2\partial\overline{\partial}f_2\right) &= e^{f_2}. \end{aligned} \tag{90}$$

Notice that there are actually just two equations as the third is just the negative of the sum of the other two.

One can then take linear combinations of these equations to obtain

$$\begin{aligned} \partial\overline{\partial}f_1 &= 2e^{f_1} - e^{f_2} \\ \partial\overline{\partial}f_2 &= -e^{f_1} + 2e^{f_2}. \end{aligned} \tag{91}$$

A.1.1 General case

For the more general case of

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}, \tag{92}$$

we need to take a more complicated unitary transformation

$$g = \begin{pmatrix} r^m e^{\frac{1}{6}(2f_1+f_2)} & 0 & 0 \\ 0 & r^n e^{\frac{1}{6}(f_2-f_1)} & 0 \\ 0 & 0 & r^{-m-n} e^{\frac{1}{6}(-f_1-2f_2)} \end{pmatrix}, \tag{93}$$

where $x = re^{i\theta}$ and the numbers m and n are determined by demanding that the second term in the D-term equation be homogeneous in r. The f_a are assumed to be independent of y and θ.

The transformed Higgs field is

$$\Phi = g\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix}g^{-1} = \begin{pmatrix} 0 & r^{m-n}e^{\frac{1}{2}f_1} & 0 \\ 0 & 0 & r^{m+2n}e^{\frac{1}{2}f_2} \\ xr^{-2m-n}e^{\frac{1}{2}(-f_1-f_2)} & 0 & 0 \end{pmatrix}. \tag{94}$$

This gives

$$\left[\Phi^\dagger, \Phi\right] = \begin{pmatrix} r^{2-4m-2n}e^{-f_1-f_2} - r^{2m-2n}e^{f_1} & 0 & 0 \\ 0 & r^{2m-2n}e^{f_1} - r^{2m+4n}e^{f_2} & 0 \\ 0 & 0 & r^{2m+4n}e^{f_2} - r^{2-4m-2n}e^{-f_1-f_2} \end{pmatrix}, \tag{95}$$

and homogeneity in r requires

$$\begin{aligned} 2 - 4m - 2n &= 2m - 2n \\ 2m - 2n &= 2m + 4n, \end{aligned} \tag{96}$$

which implies $m=\frac{1}{3},\ n=0$, and therefore the unitary transformation is

$$g=\begin{pmatrix} r^{\frac{1}{3}}e^{\frac{1}{6}(2f_1+f_2)} & 0 & 0 \\ 0 & e^{\frac{1}{6}(f_2-f_1)} & 0 \\ 0 & 0 & r^{-\frac{1}{3}}e^{\frac{1}{6}(-f_1-2f_2)} \end{pmatrix}, \tag{97}$$

and so

$$\left[\Phi^\dagger,\Phi\right]=\begin{pmatrix} r^{\frac{2}{3}}e^{-f_1-f_2}-r^{\frac{2}{3}}e^{f_1} & 0 & 0 \\ 0 & r^{\frac{2}{3}}e^{f_1}-r^{\frac{2}{3}}e^{f_2} & 0 \\ 0 & 0 & r^{\frac{2}{3}}e^{f_2}-r^{\frac{2}{3}}e^{-f_1-f_2} \end{pmatrix}. \tag{98}$$

Then, using

$$\begin{aligned} \frac{\partial}{\partial\overline{x}} &= e^{i\theta}\frac{\partial}{\partial r}+\frac{ie^{i\theta}}{r}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial x} &= e^{-i\theta}\frac{\partial}{\partial r}-\frac{ie^{-i\theta}}{r}\frac{\partial}{\partial\theta}, \end{aligned} \tag{99}$$

we derive the connection to be

$$A^{0,1}=g\overline{\partial}g^{-1}=\begin{pmatrix} -\frac{e^{i\theta}}{3r}-\frac{1}{6}\left(2\overline{\partial}f_1+\overline{\partial}f_2\right) & 0 & 0 \\ 0 & \frac{1}{6}\left(\overline{\partial}f_1-\overline{\partial}f_2\right) & 0 \\ 0 & 0 & \frac{e^{i\theta}}{3r}+\frac{1}{6}\left(\overline{\partial}f_1+2\overline{\partial}f_2\right) \end{pmatrix}, \tag{100}$$

and also

$$F_A^{1,1}=\begin{pmatrix} -\frac{1}{3}\left(2\partial\overline{\partial}f_1+\partial\overline{\partial}f_2\right) & 0 & 0 \\ 0 & \frac{1}{3}\left(\partial\overline{\partial}f_1-\partial\overline{\partial}f_2\right) & 0 \\ 0 & 0 & \frac{1}{3}\left(\partial\overline{\partial}f_1+2\partial\overline{\partial}f_2\right) \end{pmatrix}. \tag{101}$$

This means that the D-term equation gives

$$\begin{aligned} \frac{1}{3}\left(-2\partial\overline{\partial}f_1-\partial\overline{\partial}f_2\right) &= r^{\frac{2}{3}}\left(e^{f_1-f_2}-e^{f_1}\right) \\ \frac{1}{3}\left(\partial\overline{\partial}f_1-\partial\overline{\partial}f_2\right) &= r^{\frac{2}{3}}\left(e^{f_1}-e^{f_2}\right) \\ \frac{1}{3}\left(\partial\overline{\partial}f_1+2\partial\overline{\partial}f_2\right) &= r^{\frac{2}{3}}\left(e^{f_2}-e^{-f_1-f_2}\right). \end{aligned} \tag{102}$$

Again these three equations are just two independent ones, which we get after taking linear combinations and using the fact that we defined the f_a to only depend on r as:

$$\begin{aligned} \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)f_1 &= r^{\frac{2}{3}}\left(2e^{f_1}-e^{-f_1-f_2}-e^{f_2}\right) \\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)f_2 &= r^{\frac{2}{3}}\left(2e^{f_2}-e^{-f_1-f_2}-e^{f_1}\right). \end{aligned} \tag{103}$$

A.2 Details of $SU(4)$ Computations

Let us consider the spectral equation for an $SU(4)$ field:

$$P_\phi(z)=z^4-x, \tag{104}$$

for which there is a Z_4 monodromy. In the holomorphic gauge the Higgs field becomes

$$\Phi=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 \end{pmatrix}. \tag{105}$$

which is an intermediate between a diagonal Higgs field and a nilpotent Higgs field in holomorphic gauge

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{106}$$

The transition to the unitary gauge is achieved by using

$$g = \begin{pmatrix} e^{f_1} & 0 & 0 & 0 \\ 0 & e^{f_2} & 0 & 0 \\ 0 & 0 & e^{f_3} & 0 \\ 0 & 0 & 0 & e^{f_4} \end{pmatrix}, \tag{107}$$

with the unit determinant condition that $\sum_a f_a = 0$. This gives the Higgs field in unitary gauge as

$$\Phi = g \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & e^{f_1-f_2} & 0 & 0 \\ 0 & 0 & e^{f_2-f_3} & 0 \\ 0 & 0 & 0 & e^{f_3-f_4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{108}$$

the commutator part of the D-term equation is then

$$\left[\Phi^\dagger, \Phi\right] = \begin{pmatrix} -e^{2(f_1-f_2)} & 0 & 0 & 0 \\ 0 & e^{2(f_1-f_2)} - e^{2(f_2-f_3)} & 0 & 0 \\ 0 & 0 & e^{2(f_2-f_3)} - e^{2(f_3-f_4)} & 0 \\ 0 & 0 & 0 & e^{2(f_3-f_4)} \end{pmatrix}, \tag{109}$$

and to get g in terms of the linearly independent functions h_a, we require

$$\begin{aligned} 2\left(f_1 - f_2\right) &= h_1 \\ 2\left(f_2 - f_3\right) &= h_2 \\ 2\left(f_3 - f_4\right) &= h_3, \end{aligned} \tag{110}$$

along with

$$f_1 + f_2 + f_3 + f_4 = 0, \tag{111}$$

which is solved by

$$\begin{aligned} f_1 &= \frac{1}{8}\left(3h_1 + 2h_2 + h_3\right) \\ f_2 &= \frac{1}{8}\left(-h_1 + 2h_2 + h_3\right) \\ f_3 &= \frac{1}{8}\left(-h_1 - 2h_2 + h_3\right) \\ f_4 &= \frac{1}{8}\left(-h_1 - 2h_2 - 3h_3\right). \end{aligned} \tag{112}$$

The required unitary transformation g, after again writing the independent functions as f_a instead of h_a, is

$$g = \begin{pmatrix} e^{\frac{1}{8}(3f_1+2f_2+f_3)} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{8}(-f_1+2f_2+f_3)} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{8}(-f_1-2f_2+f_3)} & 0 \\ 0 & 0 & 0 & e^{\frac{1}{8}(-f_1-2f_2-3f_3)} \end{pmatrix}. \tag{113}$$

The unitary gauge Higgs field is

$$\Phi = \begin{pmatrix} 0 & e^{\frac{1}{2}f_1} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{2}f_2} & 0 \\ 0 & 0 & 0 & e^{\frac{1}{2}f_3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{114}$$

and the commutator part of the D-term equation is

$$\left[\Phi^{\dagger},\Phi\right]=\begin{pmatrix} -e^{f_1} & 0 & 0 & 0 \\ 0 & e^{f_1}-e^{f_2} & 0 & 0 \\ 0 & 0 & e^{f_2}-e^{f_3} & 0 \\ 0 & 0 & 0 & e^{f_3} \end{pmatrix}. \tag{115}$$

The unitary connection is given by

$$A^{0,1}=g\overline{\partial}g^{-1}=\frac{1}{8}\begin{pmatrix} -3\overline{\partial}f_1-2\overline{\partial}f_2-\overline{\partial}f_3 & 0 & 0 & 0 \\ 0 & \overline{\partial}f_1-2\overline{\partial}f_2-\overline{\partial}f_3 & 0 & 0 \\ 0 & 0 & \overline{\partial}f_1+2\overline{\partial}f_2-\overline{\partial}f_3 & 0 \\ 0 & 0 & 0 & \overline{\partial}f_1+2\overline{\partial}f_2+3\overline{\partial}f_3 \end{pmatrix}, \tag{116}$$

and therefore the D-term equation gives

$$\begin{aligned} \frac{1}{4}\left(-3\partial\overline{\partial}f_1-2\partial\overline{\partial}f_2-\partial\overline{\partial}f_3\right) &= -e^{f_1} \\ \frac{1}{4}\left(\partial\overline{\partial}f_1-2\partial\overline{\partial}f_2-\partial\overline{\partial}f_3\right) &= e^{f_1}-e^{f_2} \\ \frac{1}{4}\left(\partial\overline{\partial}f_1+2\partial\overline{\partial}f_2-\partial\overline{\partial}f_3\right) &= e^{f_2}-e^{f_3} \\ \frac{1}{4}\left(\partial\overline{\partial}f_1+2\partial\overline{\partial}f_2+3\partial\overline{\partial}f_3\right) &= e^{f_3}. \end{aligned} \tag{117}$$

As with $SU(3)$ the last equation is just the negative of the sum of the first three, and taking linear combinations one obtains

$$\begin{aligned} \partial\overline{\partial}f_1 &= 2e^{f_1}-e^{f_2} \\ \partial\overline{\partial}f_2 &= -e^{f_1}+2e^{f_2}-e^{f_3} \\ \partial\overline{\partial}f_3 &= -e^{f_2}+2e^{f_3}. \end{aligned} \tag{118}$$

This is of the desired form

$$\partial\overline{\partial}f_a=C_{ab}e^{f_b}, \tag{119}$$

where C_{ab} is the $SU(4)$ Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{120}$$

A.2.1 General case

For the general case, we proceed the same way as with $SU(3)$, taking a more complicated unitary transformation

$$g=\begin{pmatrix} r^{l}e^{\frac{1}{8}(3f_1+2f_2+f_3)} & 0 & 0 & 0 \\ 0 & r^{m}e^{\frac{1}{8}(-f_1+2f_2+f_3)} & 0 & 0 \\ 0 & 0 & r^{n}e^{\frac{1}{8}(-f_1-2f_2+f_3)} & 0 \\ 0 & 0 & 0 & r^{-l-m-n}e^{\frac{1}{8}(-f_1-2f_2-3f_3)} \end{pmatrix}. \tag{121}$$

Then the unitary Higgs field is

$$\Phi=g\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 \end{pmatrix}g^{-1}=\begin{pmatrix} 0 & r^{l-m}e^{\frac{1}{2}f_1} & 0 & 0 \\ 0 & 0 & r^{m-n}e^{\frac{1}{2}f_2} & 0 \\ 0 & 0 & 0 & r^{+l+m+2n}e^{\frac{1}{2}f_3} \\ xr^{-2l-m-n}e^{\frac{1}{2}(-f_1-f_2-f_3)} & 0 & 0 & 0 \end{pmatrix}. \tag{122}$$

Requiring the commutator part of the D-term equation to be homogeneous in r implies that

$$\begin{aligned} l &= \frac{3}{8} \\ m &= \frac{1}{8} \\ n &= -\frac{1}{8}, \end{aligned} \tag{123}$$

which means that the required unitary transformation is

$$g = \begin{pmatrix} r^{\frac{3}{8}}e^{\frac{1}{8}(3f_1+2f_2+f_3)} & 0 & 0 & 0 \\ 0 & r^{\frac{1}{8}}e^{\frac{1}{8}(-f_1+2f_2+f_3)} & 0 & 0 \\ 0 & 0 & r^{-\frac{1}{8}}e^{\frac{1}{8}(-f_1-2f_2+f_3)} & 0 \\ 0 & 0 & 0 & r^{-\frac{3}{8}}e^{\frac{1}{8}(-f_1-2f_2-3f_3)} \end{pmatrix}, \tag{124}$$

and

$$\left[\Phi^\dagger, \Phi\right] = \begin{pmatrix} r^{\frac{1}{2}}e^{-f_1-f_2-f_3} - r^{\frac{1}{2}}e^{f_1} & 0 & 0 & 0 \\ 0 & r^{\frac{1}{2}}e^{f_1} - r^{\frac{1}{2}}e^{f_2} & 0 & 0 \\ 0 & 0 & r^{\frac{1}{2}}e^{f_2} - r^{\frac{1}{2}}e^{f_3} & 0 \\ 0 & 0 & 0 & r^{\frac{1}{2}}e^{f_3} - r^{\frac{1}{2}}e^{-f_1-f_2-f_3} \end{pmatrix}. \tag{125}$$

The connection is

$$\frac{1}{8}\begin{pmatrix} -\frac{3e^{i\theta}}{r} - 3\overline{\partial}f_1 - 2\overline{\partial}f_2 - \overline{\partial}f_3 & 0 & 0 & 0 \\ 0 & -\frac{e^{i\theta}}{r} + \overline{\partial}f_1 - 2\overline{\partial}f_2 - \overline{\partial}f_3 & 0 & 0 \\ 0 & 0 & \frac{e^{i\theta}}{r} + \overline{\partial}f_1 + 2\overline{\partial}f_2 - \overline{\partial}f_3 & 0 \\ 0 & 0 & 0 & \frac{3e^{i\theta}}{r} + \overline{\partial}f_1 + 2\overline{\partial}f_2 + 3\overline{\partial}f_3 \end{pmatrix}, \tag{126}$$

and so the D-term equation is

$$\begin{aligned} \frac{1}{4}\left(-3\partial\overline{\partial}f_1 - 2\partial\overline{\partial}f_2 - \partial\overline{\partial}f_3\right) &= r^{\frac{1}{2}}\left(e^{-f_1-f_2-f_3} - e^{f_1}\right) \\ \frac{1}{4}\left(\partial\overline{\partial}f_1 - 2\partial\overline{\partial}f_2 - \partial\overline{\partial}f_3\right) &= r^{\frac{1}{2}}\left(e^{f_1} - e^{f_2}\right) \\ \frac{1}{4}\left(\partial\overline{\partial}f_1 + 2\partial\overline{\partial}f_2 - \partial\overline{\partial}f_3\right) &= r^{\frac{1}{2}}\left(e^{f_2} - e^{f_3}\right) \\ \frac{1}{4}\left(\partial\overline{\partial}f_1 + 2\partial\overline{\partial}f_2 + 3\partial\overline{\partial}f_3\right) &= r^{\frac{1}{2}}\left(e^{f_3} - e^{-f_1-f_2-f_3}\right). \end{aligned} \tag{127}$$

As with the nilpotent case, the fourth equation here is just the negative sum of the first three. Taking linear combinations, one obtains

$$\begin{aligned} \partial\overline{\partial}f_1 &= r^{\frac{1}{2}}\left(2e^{f_1} - e^{f_2} - e^{-f_1-f_2-f_3}\right) \\ \partial\overline{\partial}f_2 &= r^{\frac{1}{2}}\left(-e^{f_1} + 2e^{f_2} - e^{f_3}\right) \\ \partial\overline{\partial}f_3 &= r^{\frac{1}{2}}\left(-e^{f_2} + 2e^{f_3} - e^{-f_1-f_2-f_3}\right). \end{aligned} \tag{128}$$

A.3 General $SU(n)$

Here we break $SU(n)$ with an $n \times n$ matrix-valued Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{129}$$

A.3.1 Nilpotent case ($x = 0$)

The nilpotent Higgs field in holomorphic gauge is

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{130}$$

To solve the D-term equation, as with $SU(3)$ and $SU(4)$, we move to unitary gauge with a transformation

$$g = \begin{pmatrix} e^{f_1} & 0 & \cdots & 0 \\ 0 & e^{f_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{f_n} \end{pmatrix}, \tag{131}$$

where the unit determinant condition for g implies $\sum f_a = 0$.

The most convenient way to express f_a in terms of the $n-1$ linearly independent h_a's is

$$f_1 = \frac{1}{2n}\left((n-1)\,h_1 + (n-2)\,h_2 + \cdots 2h_{n-2} + h_{n-1}\right). \tag{132}$$

With the rest of the f_a determined by the conditions

$$2\left(f_a - f_{a+1}\right) = h_a. \tag{133}$$

With this particular unitary transformation, the D-term equation simply becomes

$$\partial\overline{\partial} h_a = C_{ab} e^{h_b}, \tag{134}$$

where C_{ab} is the Cartan matrix of $SU(n)$.

A.3.2 General case

For the general case, as before, a more complicated unitary transformation is required:

$$g = \begin{pmatrix} r^{m_1} e^{f_1} & 0 & \cdots & 0 \\ 0 & r^{m_2} e^{f_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r^{m_n} e^{f_n} \end{pmatrix}, \tag{135}$$

with $\sum m_a = 0$ and the f_a defined as in equations (132) and (133). The unitary Higgs field is then

$$\Phi = \begin{pmatrix} 0 & r^{m_1-m_2} e^{\frac{1}{2}h_1} & 0 & \cdots & 0 \\ 0 & 0 & r^{m_2-m_3} e^{\frac{1}{2}h_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r^{m_{n-1}-m_n} e^{\frac{1}{2}h_{n-1}} \\ x r^{m_n-m_1} e^{-\frac{1}{2}\left(\sum h_a\right)} & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{136}$$

and the commutator part of the D-term equation is

$$\begin{pmatrix} r^{2+2m_n-2m_1} e^{-\sum h_a} - r^{2m_1-2m_2} e^{h_1} & 0 & \cdots & 0 \\ 0 & r^{2m_1-2m_2} e^{h_1} - r^{2m_2-2m_3} e^{h_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r^{2m_{n-1}-2m_n} e^{h_{n-1}} - r^{2+2m_n-2m_1} e^{-\sum h_a} \end{pmatrix}. \tag{137}$$

The D-term can then be made homogeneous in r by choosing

$$m_a = \frac{n+1-2a}{2n}. \tag{138}$$

Then D-term equation then becomes

$$\begin{aligned}
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) h_1 &= r^{\frac{2}{n}} \left(-e^{-\sum h_a} + 2e^{h_1} - e^{h_2}\right) \\
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) h_a &= r^{\frac{2}{n}} C_{ab} e^{h_b} \qquad a = 2, \ldots, n-2 \\
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) h_{n-1} &= r^{\frac{2}{n}} \left(-e^{h_{n-2}} + 2e^{h_{n-1}} - e^{-\sum h_a}\right).
\end{aligned} \tag{139}$$

References

[1] Ron Donagi and Martijn Wijnholt. "Model Building with F-Theory". 2008. arXiv:hep-th/0802.2969.

[2] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - I". *JHEP*, 0901:058, 2009. arXiv:hep-th/0802.3391.

[3] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions". *JHEP*, 0901:059, 2009. arXiv:hep-th/0806.0102.

[4] Radu Tatar and Taizan Watari. "GUT Relations from String Theory Compactifications". *Nucl.Phys.*, B810:316–353, 2009. arXiv:hep-th/0806.0634.

[5] Radu Tatar, Yoichi Tsuchiya, and Taizan Watari. "Right-handed Neutrinos in F-theory Compactifications". *Nucl.Phys.*, B823:1–46, 2009. arXiv:hep-th/0905.2289.

[6] Hirotaka Hayashi, Radu Tatar, Yukinobu Toda, Taizan Watari, and Masahito Yamazaki. "New Aspects of Heterotic–F Theory Duality". *Nucl.Phys.*, B806:224–299, 2009. arXiv:hep-th/0805.1057.

[7] Cumrun Vafa. "Geometry of Grand Unification". 2009. arXiv:math-ph/0911.3008.

[8] Jonathan J. Heckman. "Particle Physics Implications of F-theory". *Ann.Rev.Nucl.Part.Sci.*, 2010. arXiv:hep-th/1001.0577.

[9] Timo Weigand. "Lectures on F-theory compactifications and model building". *Class.Quant.Grav.*, 27:214004, 2010. arXiv:hep-th/1009.3497.

[10] Hirotaka Hayashi, Teruhiko Kawano, Radu Tatar, and Taizan Watari. "Codimension-3 Singularities and Yukawa Couplings in F-theory". *Nucl.Phys.*, B823:47–115, 2009. arXiv:hep-th/0901.4941.

[11] Sergio Cecotti, Clay Cordova, Jonathan J. Heckman, and Cumrun Vafa. "T-Branes and Monodromy". 2010. arXiv:hep-th/1010.5780.

[12] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, and Taizan Watari. "More on Dimension-4 Proton Decay Problem in F-theory – Spectral Surface, Discriminant Locus and Monodromy". *Nucl.Phys.*, B840:304–348, 2010. arXiv:hep-th/1004.3870.

[13] Jonathan J. Heckman, Alireza Tavanfar, and Cumrun Vafa. "The Point of E(8) in F-theory GUTs". *JHEP*, 1008:040, 2010. arXiv:hep-th/0906.0581.

[14] Alon E. Faraggi. "Proton stability in superstring derived models". *Nucl.Phys.*, B428:111–125, 1994. arXiv:hep-ph/9403312.

[15] Alon E. Faraggi. "Proton stability and superstring Z-prime". *Phys.Lett.*, B499:147–157, 2001. arXiv:hep-ph/0011006.

[16] Jogesh C. Pati. "The Essential role of string derived symmetries in ensuring proton stability and light neutrino masses". *Phys.Lett.*, B388:532–542, 1996. arXiv:hep-ph/9607446.

[17] Claudio Coriano, Alon E. Faraggi, and Marco Guzzi. "A Novel string derived Z-prime with stable proton, light-neutrinos and R-parity violation". *Eur.Phys.J.*, C53:421–428, 2008. arXiv:hep-ph/0704.1256.

[18] Sheldon H. Katz and Cumrun Vafa. Matter from geometry. *Nucl.Phys.*, B497:146–154, 1997. arXiv:hep-th/9606086.

[19] S. Katz and D. R. Morrison. "Gorenstein Threefold Singularities with Small Resolutions via Invariant Theory for Weyl Groups". 1992. arXiv:alg-geom/9202002.

[20] Vincent Bouchard, Jonathan J. Heckman, Jihye Seo, and Cumrun Vafa. "F-theory and Neutrinos: Kaluza-Klein Dilution of Flavor Hierarchy". *JHEP*, 1001:061, 2010. arXiv:hep-ph/0904.1419.

[21] Joseph Marsano, Natalia Saulina, and Sakura Schafer-Nameki. "Compact F-theory GUTs with U(1) (PQ)". *JHEP*, 1004:095, 2010. arXiv:hep-th/0912.0272.

[22] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, and Taizan Watari. "Flavor Structure in F-theory Compactifications". *JHEP*, 08:036, 2010. arXiv:hep-th/0910.2762.

[23] Emilian Dudas and Eran Palti. "Froggatt-Nielsen models from E8 in F-theory GUTs". *JHEP*, 01:127, 2010. arXiv:hep-th/0912.0853.

[24] Joseph Marsano, Natalia Saulina, and Sakura Schafer-Nameki. "Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs". *JHEP*, 0908:046, 2009. arXiv:hep-th/0906.4672.

[25] Keith R. Dienes and John March-Russell. "Realizing higher level gauge symmetries in string theory: New embeddings for string GUTs". *Nucl.Phys.*, B479:113–172, 1996. arXiv:hep-th/9604112.

[26] Keith R. Dienes. "New constraints on SO(10) model building from string theory". *Nucl.Phys.*, B488:141–158, 1997. arXiv:hep-ph/9606467.